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Relation Between Strings and Ribbon Knots

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A ribbon knot can be represented as the propagation of an open string in (Euclidean) space-time. By imposing physical conditions plus an ansatz on the string scattering amplitude, we get invariant polynomials of ribbon knots which correspond to Jones and Wadati *et al.* polynomials for ordinary knots. Motivated by the string scattering vertices, we derive an algebra which is a generalization of Hecke and Murakami-Birman-Wenzel (BMW) algebras of knots.

String theory (Green *et al.* 1987) is a promising candidate for both quantum gravity and unified field theory. Its main promise is in the rich mathematical structure it has. It is related to the theory of Riemann surfaces, algebraic and differential topology, and conformal field theory. Recently it has been related to topological quantum field theory in three dimensions (Witten, 1989). In this paper it will be related to the theory of ribbon knots (Kauffman, 1987*a*).

Recalling that the world line of a point particle is a braid whose closure is a knot, the world line of an open string is a ribbon braid whose closure is a ribbon knot. For ribbon knots one has the extra freedom of twist. We will not attempt to prove an analogue to Alexander's theorem (Kauffman, 1987*a*), i.e., every ribbon knot corresponds to a ribbon braid. We study only the class of ribbon knots which can be constructed via the closure of ribbon braids (up to twist).

It is important to realize that the correspondence between ribbon knots and one-dimensional knots is not one-to-one. As an example, it is easy to

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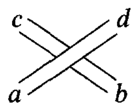
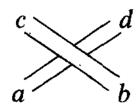
see that both a closed ribbon and a Möbius strip correspond to the one-dimensional unknot, although they are topologically different. Therefore the problem of constructing ribbon-knot-invariant polynomials is important.

Recently Kauffman introduced a Feynman-type approach to constructing one-dimensional knot invariants (Kauffman, 1987*b*). His work has been extended (Ge *et al.*, 1989) to the $N=3$ and 4 cases of Wadati *et al.* polynomials (Wadati *et al.*, 1989). Here we use this technique to build invariant polynomials for ribbon knots.

The basic idea is to consider the crossings of ribbon knots as scattering of open strings. Imposing physically plausible constraints, e.g., charge conservation and *PCT* invariance, and assuming an ansatz for the scattering amplitude, one gets the required invariants. We assign charges to each string which change only at the crossings (vertices). They take the values $\{(-N+1)/2, (-N+3), \ldots, (N-1)/2\}$ for some integer N.

The scattering matrix S_{ab}^{cd} denotes the 4-point scattering

$S_{ab}^{cd} =$ 

$(S^{-1})_{ab}^{cd} =$ 

(1)

Unitarity and factorizability conditions are

$$S_{ab}^{cd}(S^{-1})_{cd}^{ef} = \delta_a^e \delta_b^f \tag{2}$$

$$S_{ab}^{a'b'} S_{b'c}^{c'f} S_{a'c'}^{de} = S_{ab'}^{da'} S_{bc}^{b'c'} S_{a'c'}^{ef} \tag{3}$$

where one sums over repeated indices. Equation (3) is the Yang–Baxter equation. For $N=2$ we follow Kauffman by imposing the ansatz

$$S_{ab}^{cd} = \begin{cases} t - t^{-1} & a=c<b=d \\ t, & a=c=b=d \\ 1, & a=d\neq b=c \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

This ansatz satisfies the following physical constraints:

(i) Charge conservation

$$S_{ab}^{cd} = 0 \qquad \text{unless} \quad a+b=c+d \tag{5}$$

(ii) *PC* invariance

$$S_{ab}^{cd} = S_{-b-a}^{-d-c} \tag{6}$$

(iii) *T* invariance

$$S_{ab}^{cd} = S_{cd}^{ab} \tag{7}$$

Conditions (6) and (7) imply *PCT* invariance

$$S_{ab}^{cd} = S_{-d-c}^{-b-a} \tag{8}$$

Apart from the previous constraints, there is no general way to derive the ansatz (4) yet.

Now we obtain the analogue of the Kauffman bracket polynomial for ribbon knots in the form

$$[K] = \sum_{S} \langle S \mid K \rangle t^{-|S|} \tag{9}$$

where S is a choice of decomposition for all the vertices in the ribbon knot K. Here $\langle S \mid K \rangle$ is the multiplication of all the factors corresponding to such a decomposition. Since S ends up in a set of closed ribbons, $|S|$ is defined by

$$|S| = \sum_{l\text{-closed ribbons}} \varepsilon_l \times (\text{change of the ribbon } l) \tag{10}$$

where $\varepsilon_l = +1$ (-1) according to whether the charge orientation follows the right (left)-hand rule with respect to the closed ribbon axis.

Notice that the bracket polynomial (9) is regular isotopy invariant and not ambient isotopy invariant. Regular isotopy, however, seems more natural from the point of view of topological quantum field theory (Witten, 1989).

The $N = 3$ case is obtained using the following ansatz, which is analogous to the corresponding case for the one-dimensional knot (Ge *et al.*, 1989)

$$S_{11}^{11} = S_{-1-1}^{-1-1} = t^2, \qquad S_{00}^{00} = 1$$

$$S_{01}^{01} = S_{-10}^{-10} = t^2 - t^{-2} \tag{11}$$

$$S_{-11}^{-11} = t^2 - 1 - t^{-2} + t^{-4}$$

$$S_{0-1}^{-10} = S_{10}^{01} = S_{01}^{10} = S_{-10}^{0-1} = -1$$

$$S_{1-1}^{-11} = S_{-11}^{1-1} = T^{-2} \tag{12}$$

$$S_{00}^{-11} = S_{-11}^{00} = t - t^{-3}$$

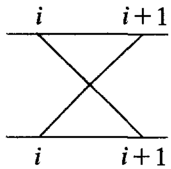
Substituting in (9), one gets the $N = 3$ regular isotopy-invariant polynomial for ribbon knots.

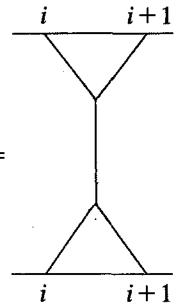
It is important to notice that the polynomials constructed here do not distinguish the twist number; therefore, they are not the most general invariant polynomials for ribbon knots.

The world line of a closed string is a vein braid whose closure forms a vein knot, which will not be considered here.

We will discuss the effect of strings on the algebra of knots. So far, all the studies of knots use the 4-valent vertices, which correspond to the 4-point scattering amplitude. This is unsatisfactory from the point of view of strings, since it is known that strings have both 3-point and 4-point scatterings. Furthermore, one can form the 4-point scattering out of two 3-point ones, as is known in the Fermi theory of weak interactions.

Motivated by the previous argument, we propose the following algebra generated by $\{b_i, e_i\}$, where $i = 1, 2, \ldots, n-1$, where

$b_i =$ 

$e_i =$ 

(13)

The multiplication is defined, as usual, by concatenation, erasing the middle line and rescaling. A factor α is multiplied for each closed loop and a factor β is multiplied for each twist. Then it is straightforward to derive the following defining relations for the algebra:

$$\begin{aligned} b_i b_j &= b_j b_i \qquad \text{if} \quad |i-j| \geq 2 \\ b_i b_{i+1} b_i &= b_{i+1} b_i b_{i+1} \\ e_i^2 &= \alpha e_i \\ e_i b_i &= b_i e_i = \beta e_i \end{aligned} \tag{14}$$

This algebra contains both the Hecke algebra (Jones, 1987) and the Murakami–Birman–Wenzel algebra (Murakami, 1987; Birman and Wenzel, 1989), which correspond to Jones polynomials and $N = 3$ Wadati *et al.* polynomials for knots, respectively.

An interesting way of looking at the corresponding to knots is to look at the decomposition of the scattering amplitude. For $N = 2$ the decomposition of the scattering amplitude (4) corresponds to the Hecke algebra. For $N = 3$ the decomposition (11) corresponds to the Murakami–Birman–Wenzel algebra. For $N = 4$ the algebra is not known yet despite the discovery of the decomposition of the scattering amplitude (Ge *et al.*, 1989). The decomposition is rather complicated, so we will not write it here. From the graphical point of view it is similar to that of the $N = 3$ case plus terms which can be represented by the operator e_i of the algebra (14). Therefore we expect that the algebra (14) will be related to the as yet unknown $N = 4$ algebra. This problem is currently under investigation.

Invariant polynomials of ribbon knots corresponding to $N = 2$ and $N = 3$ polynomials for ordinary (one-dimensional knots) have been constructed. They are not the most general invariants of ribbon knots, since they do not distinguish the twist character. Motivated by the string 3-point interaction, the algebra (14) is proposed. It is a generalization of both the $N = 2$ (Hecke) and $N = 3$ (BMW) algebras. We anticipate that this algebra will help in solving the outstanding problem of $N = 4$ algebra.

REFERENCES

Birman, J., and Wenzel, H. (1989). *Transactions of the American Mathematical Society*, **313**, 249.

Ge, M., Wang, L., Xue, K., and Wu, Y. (1989). In *Braid Group, Knot Theory and Statistical Mechanics*, C. N. Yang and M. Ge, eds., World Scientific, Singapore.

Green, M., Schwarz, J. H., and Witten, E. (1987). *Superstring Theory*, Cambridge University Press, Cambridge.

Jones, V. F. R. (1987). *Annals of Mathematics*, **126**, 335.

Kauffman, L. H. (1987*a*). *On Knots*, Princeton University Press, Princeton, New Jersey.

Kauffman, L. H. (1987*b*). *Topology*, **26**, 395.

Murakami, J. (1987). *Osaka Journal of Mathematics*, **24**, 745.

Wadati, M., Deguchi, T., and Akutsu, Y. (1989). *Physics Reports*, **180**, 247.

Witten, E. (1989). *Communications in Mathematical Physics*, **121**, 351.